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Analytical solutions of 1-D heat conduction problem for a single fin with temperature dependent heat transfer coefficient – I. Closed-form inverse solution

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Abstract

Closed-form solution of 1-D heat conduction problem for a single straight fin and spine of constant cross-section has been obtained. The local heat transfer coefficient is assumed to vary as a power function of temperature excess. The dependence of the fin parameter N on the dimensionless temperature difference T_e at the fin tip for a given exponent n was derived in a form $N/N_0 = T_e^{-\mu n}$ (where N_0 is a well-known N expression for $n = 0$). Coefficient μ was found to be equal to 5/12 according to the exact solution at $T_e \to 1$ or to 0.4 according to the fitting procedure for the data of the numerical integration. Obtained formula serves as a basis for the derivation of the direct expressions for T_e vs N at given n, fin base thermal conductance and augmentation factor presented in the second part of the study. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

Heat conduction problems for the fins and finned surfaces with a non-uniform heat transfer coefficient along a fin have been considered for a long time in the special literature (see, for instance, a book by Kern and Kraus [1] and a comprehensive review by Kraus [2]). The problems related to a dependence of the heat transfer coefficient h on the spatial coordinate only along a fin have been discussed in [1]. This type of problems are typical for the fins and extended surfaces with a forced convection heat transfer. The dependence of the heat transfer coefficient on the local temperature difference ϑ between a fin surface and surrounding medium is appropriate to such kinds of heat transfer as natural convection, radiation and, especially, boiling. This dependence is expressed mainly in a power law-type form as $h = a\vartheta^n$, where exponent n depends on the heat transfer mode whereas the coefficient a is defined by physical properties of the surrounding medium. This type of dependence is widely used to solve the heat conduction problems for a single fin beginning from the paper by Lai and Hsu [3]. The similar dependence was also used by Dul'kin et al. [4] and Petukhov et al. [5]. In papers [3–5] a boiling heat transfer on a single fin has been considered taking into account stable coexistence of the different heat transfer modes on its surface guided by the mentioned power law-type dependence on local temperature excess of the fin surface. The 1-D heat conduction problem for a single fin has been solved analytically and verified by experimental data. All these papers have been inspired by preceding works by Cumo et al. [6] and Haley and Westwater [7], where the very problem has been solved in 2-D and 1-D statement, respectively, using numerical methods. These solutions and other contributions, certain of which were not included in the paper by Kraus [2], have been considered in detail in a book by Roizen and Dul'kin [8]. In the last half of the 1980s and first half of the 1990s some researchers from the USA [11], Dutch [12–15] and Taiwan [16,17] undertook extensive studies of this type problems. Their contribution will be considered below after a short introduction to the problem.

The temperature distribution along a fin and the thermal conductance at the fin base can be determined

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Nomenclature

A cross-section area of the fin (m^2)
a given constant in Eq. (1) ($W\ m^{-2}\ K^{-(n+1)}$)
$B(n)$ fitting coefficient in Eq. (19) depending on n
b_1, b_2 variables which depend on exponent n and indicate whether the integral reduces to a simple one of rational function
F hypergeometric function defined in the introduction
h heat transfer coefficient ($W\ m^{-2}\ K^{-1}$)
h_b heat transfer coefficient for the fin base ($W\ m^{-2}\ K^{-1}$)
i number of the split elements for interval $1 - T_e$
k thermal conductivity of the fin material ($W\ m^{-1}\ K^{-1}$)
l fin height (m)
n given exponent in Eq. (1)
N dimensionless fin parameter defined by Eq. (4)
N^+ fin parameter corresponding to $T_e = 0$ for $-1 \leqslant n < 0$
N^* fin parameter corresponding to maximum of the N vs T_e curve at $n < -1$
P circumference of the fin cross-section area (m)
t temperature of the fin (°C)
t_a temperature of the ambient fluid surrounding the fin (°C)
T dimensionless temperature excess of the fin
T_e dimensionless temperature difference between the fin tip and ambient fluid
T_e^* dimensionless temperature difference between the fin tip and ambient fluid corresponding to N^* for $n < -1$
x space coordinate (m)
X dimensionless space coordinate defined in the text;

Greek symbols
δ thickness of the fin (m)
ϵ length of a small split element in Eq. (15); small parameter in Eqs. (B.5), (B.6)
ϑ temperature difference between a fin and ambient fluid (°C)
ξ current integration variable
φ relative deviation (%)
ψ third parameter in hypergeometric function;

Subscripts and superscripts
a refers to the ambient fluid
b refers to the fin base (i.e. $x = l$ and $X = 1$)
e refers to the fin tip (i.e. $x = 0$ and $X = 0$)

by a solution of the 1-D heat conduction problem for a fin. Such a solution named direct allows to calculate a heat flow rate through the fin and its augmentation factor for given values of the fin dimensions (height l, cross-section area A and circumference P), heat transfer law (a and n) and fin base temperature difference ϑ_b. Unfortunately, in general case, the solution of this type problems is obtained in a form of dimensionless spatial coordinate $X = x/l$ vs current dimensionless temperature difference $T = \vartheta/\vartheta_b$ for a given parameter n, i.e., in a so-called "inverse" form. The spatial coordinate X, is expressed through an improper definite integral with respect to T for a given parameter n. At $n \neq -2$ this solution can be written as follows:

$$X \cdot N = \int_{T_e}^{T} \mathrm{d}T \Big/ \sqrt{[2/(n+2)]\left(T^{n+2} - T_e^{n+2}\right)},$$

where $N = l\sqrt{a\vartheta_b^n P/(kA)}$ is the convective–conductive parameter of a fin. Similar expression for the whole fin has a form

$$N = \int_{T_e}^{1} \mathrm{d}T \Big/ \sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})},$$

where $T_e = \vartheta/\vartheta_b$ is the dimensionless temperature difference at the fin tip. As it was pointed out in [3] this type integrals "generally cannot be evaluated in terms of simple well-known functions". Later Ünal [12–14] has shown that they can be integrated analytically in a closed form only for certain distinct values of n. Introducing variables $b_1 = 1/(n+2)$ and $b_2 = -n/(n+2)$ he has demonstrated that if either b_1 or b_2 is a positive or negative integer or zero, the integration of the above expressions can be reduced to a simple integral of the rational function. The latter can be integrated in a form of ordinary (i.e., algebraic, logarithmic and circular) functions. If neither b_1 nor b_2 is an integer or zero, the integrals in the above equations cannot be expressed by ordinary functions in accordance with a theorem proved by Tchebichev [18]. Therefore, the result of the integration can be usually expressed in a form of special functions (see, for instance, [13,14]). In this case all expressions are implicit with respect to T_e and their inversion into an explicit form is possible only for distinct values of n. For example, the above integral is reduced to algebraic formulae at $n = -4$, $n = -1$ and to a well-known hyperbolic closed-form formula at $n = 0$. For $n = 2$ this integral is an elliptic integral of the first kind but the relation between T_e, N and n remains implicit. According to [3], "for $n = 3$, 4, etc. the integral equation would yield hyperelliptic integrals and thus would involve hyperelliptic functions. However, for arbitrary value of n (for example, between 2 and 4), only numerical or graphical integration can be used ...". For negative n Ünal [12] yields 11 values between $n = -4$ and $n = -1$, for which above equations can be practically integrated. He mentioned also that "for other values of n, including $n = -2$, the use of a numerical method seems almost inevitable, at least, for practical applications". Indeed, the dependence of T_e on N at given n, can be always determined numerically, for example, by Simpson method. Later, Sen and Trinh [11] and Liaw and Yeh [16,17] following the studies by Mehta and Aris [9,10] have shown that, in general case, the above integral can be taken exactly analytically in form of a three-parametrical hypergeometric function $F(a, b, c, \psi)$, i.e.,

$$N = \int_{T_e}^{1} \mathrm{d}T \Big/ \sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})} = \sqrt{\frac{2}{n+2}\psi}\, F\left[1; \frac{n+1}{n+2}; \frac{3}{2}; \psi\right],$$

where $a = 1$ and $c = 3/2$ are numerical parameters, $b = (n + 1)/(n + 2)$ is parameter depending on n only and $\psi = 1 - T_e^{n+2}$ is parameter depending on T_e and n. The integral equation (13) for $n = -2$ (see below) can be also taken analytically in a form of a special function (Dawson's integral [16]). Unfortunately, the hypergeometric function and Dawson's integral are implicit expressions relative T_e as well. To find the fin tip temperature excess T_e at given values of the fin parameter N and the exponent n, the integral or the hypergeometric function should be considered as implicit equation relative T_e which can be solved only by iterations. Such calculations are acceptable in scientific investigations but are inconvenient for engineering design and optimization. As can be quoted from a paper by Snider and Kraus [19], "for the design engineer, the desirability of simple closed form expressions may well outweigh considerations of rigor and exactness".

Thus, the first part of our study has the objective to determinate the explicit solution for the mentioned integral expressed in terms of ordinary functions to describe the dependence N on T_e for arbitrary value of exponent n (negative or positive). Such a solution has been derived using a tangent to the curve N/N_0 vs T_e at $T_e \to 1$ (where N is an exact solution in a form of the above-mentioned hypergeometric function and N_0 is the well-known expression for N at $n = 0$). To check the applicability of our results in a more wide range of T_e, another approach is also used. The fitting procedure has been performed to approximate the data of the numerical integration of the dependence N/N_0 on T_e in log–log scale. The applicability range of the obtained solution and its relative deviation from the results of the numerical integration are considered.

Direct expressions to calculate T_e for given values of n and N, the fin base thermal conductance and its augmentation factor (effectiveness) can be derived with the obtained closed-form equation (see the second part of the study [20]). Thus, the obtained closed-form equation in terms of the ordinary functions can serve as a basis for the simple thermal design of the fins and finned surfaces with heat transfer coefficient depending on temperature excess.

2. Theoretical analysis

We will solve a heat conduction problem for a straight fin of the rectangular profile (a longitudinal fin or a spine) with an uniform cross-section area A of arbitrary form having a circumference P. The local heat transfer coefficient along the fin surface is considered to exhibit a power law-type dependence on the local temperature difference between the fin and the ambient fluid, i.e.,

$$h = a(t - t_a)^n = a\vartheta^n. \quad (1)$$

As in [16], an one-dimensional equation of the steady-state heat conduction for a fin is analysed. This equation in terms of dimensionless variables $X = x/l$ and $T = (t - t_a)/(t_b - t_a) = \vartheta/\vartheta_b$ can be written as

$$\frac{\mathrm{d}^2 T}{\mathrm{d}X^2} - N^2 T^{n+1} = 0, \quad (2)$$

where the convective–conductive parameter of the fin, N, is defined as

$$N = l\sqrt{\frac{h_b P}{kA}} = l\sqrt{\frac{a\vartheta_b^n P}{kA}}. \quad (3)$$

In the above equation h_b, l, A, P represent the heat transfer coefficient at the fin base defined according to Eq. (1) by the temperature difference ϑ_b, the fin height, area and circumference of the fin cross-section, respectively, k is the conductivity of the fin material.

Boundary conditions to Eq. (2) can be expressed as follows:

$$\frac{\mathrm{d}T}{\mathrm{d}X} = 0 \ \ (T = T_e), \quad X = 0, \quad (4)$$

$$T = 1, \quad X = 1. \quad (5)$$

To avoid an increase of the parameters number, only the case of a fin with an insulated tip (boundary condition Eq. (4)) is considered in this article. A role of the fin tip heat transfer has been comprehensively covered in [15].

Taking into account the boundary condition (Eq. (4)), the first integration of Eq. (2) from a tip of the fin ($X = 0$ and respectively $T = T_e$) to a current fin coordinate X (and respectively to a current dimensionless temperature difference T) yields

$$dT/dX = N\sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})}, \quad n \neq -2, \quad (6)$$

$$dT/dX = N\sqrt{2\ln(T/T_e)}, \quad n = -2. \quad (7)$$

Separating variables in Eqs. (6) and (7) and integrating between the limits $X = 0$ and $X = 1$ ($T = T_e$ and $T = T_b$, respectively) we get the following expressions for dimensionless temperature distribution along a fin

$$X \cdot N = \int_{T_e}^{T} dT \Big/ \sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})}, \quad n \neq -2, \quad (8)$$

$$X \cdot N = \int_{T_e}^{T} dT \Big/ \sqrt{2\ln(T/T_e)}, \quad n = -2. \quad (9)$$

As a main integral parameter of the fin, we use its thermal conductance at the base (an "input" fin thermal conductance). It is equal to the fin base heat transfer rate divided by the corresponding temperature difference $g_b = Q_b/\vartheta_b$. The quotient of g_b and the thermal conductance of the same fin but with a thermal insulation of the lateral surface $g_l = kA/l$ gives a dimensionless thermal conductance of the fin at base $G_b = (Q_b/\vartheta_b)(l/kA)$. Introducing boundary condition Eqs. (5)–(7) we get

$$G_b = dT/dX|_{X=1} = N\sqrt{[2/(n+2)](1 - T_e^{n+2})}, \quad n \neq -2, \quad (10)$$

$$G_b = dT/dX|_{X=1} = N\sqrt{2\ln(1/T_e)}, \quad n = -2. \quad (11)$$

Eqs. (10) and (11) show that a dimensionless thermal conductance at the fin base G_b depends on the given values of the fin parameter N and exponent n and on an preliminary unknown value of the dimensionless temperature difference at the fin tip T_e. To find relation between the latter value and the fin parameter N, Eqs. (6) and (7) should be integrated by the method of dividing variables in the limits $X = 0$ and $X = 1$ ($T = T_e$ and $T = 1$, respectively). As a result we get

$$N = \int_{T_e}^{1} dT \Big/ \sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})}, \quad n \neq -2, \quad (12)$$

$$N = \int_{T_e}^{1} dT \Big/ \sqrt{2\ln(T/T_e)}, \quad n = -2. \quad (13)$$

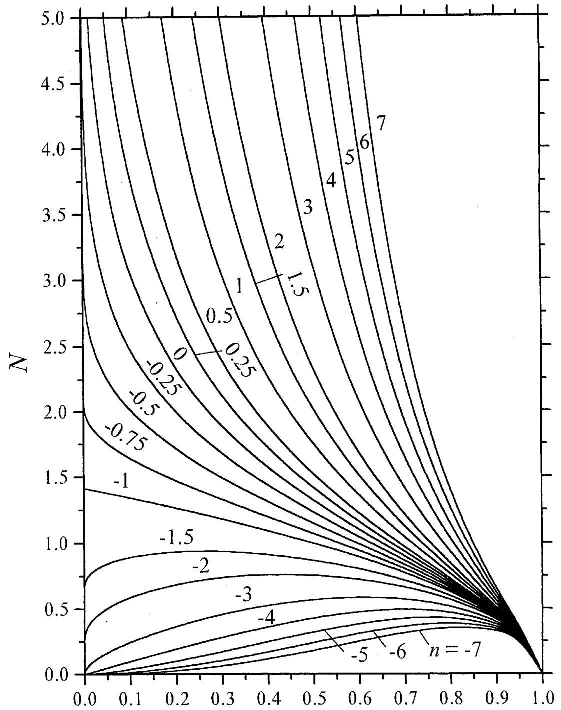

Fig. 1. The set of curves N vs T_e calculated by the numerical integration of Eqs. (12) and (13).

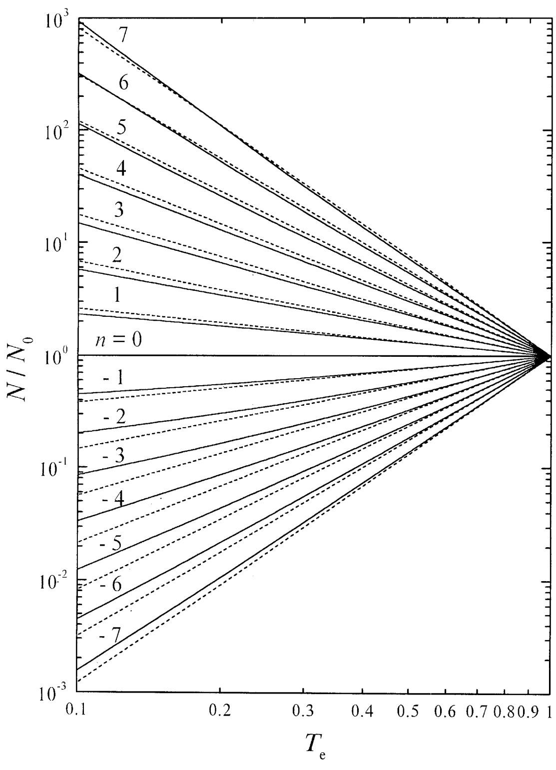

Fig. 2. The set of curves N/N_0 vs T_e in log–log scale calculated by the numerical integration of Eqs. (12) and (13) (solid lines) and with formula (18) (short dashed straight lines) for different values of n.

2.1. Numerical evaluation of the N vs T_e curves for given n

The improper definite integral in Eq. (12) has been determined numerically by its partitioning on two integrals to isolate singularity at the bottom limit of integration

$$N = \int_{T_e}^{1} \mathrm{d}T \Big/ \sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})} = \int_{T_e}^{T_e+\epsilon} \mathrm{d}T \Big/ \sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})} + \int_{T_e+\epsilon}^{1} \mathrm{d}T \Big/ \sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})}. \quad (14)$$

The first integral in RHS of Eq. (14) can be taken using a change of the variable (see Appendix A) that gives

$$\int_{T_e}^{T_e+\epsilon} \mathrm{d}T \Big/ \sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})} = \sqrt{2\epsilon/T_e^{n+1}}, \quad (15)$$

where $\epsilon = (1 - T_e)/i$ is the length, and i is the number of the split elements for the interval $1 - T_e$. The second integral in RHS of Eq. (14) can be taken by the ordinary Simpson method. The relative error of the integral evaluation performed in a range of $10^{-4} \leqslant T_e \leqslant 0.995$ and $-7 \leqslant n \leqslant 7$ does not exceed $\pm 0.14\%$ at $i = 10^4$.

The dependence of N on T_e calculated by means of Eq. (14) for $-7 \leqslant n \leqslant 7$ in the practically used range $0 < N \leqslant 5$ is displayed in Fig. 1. This dependence is shown to be appreciably non-linear and varies qualitatively different for three ranges of exponent n. The curves for all n start from the point $T_e = 1$, $N = 0$. For negative $n < -1$ the dependence of N on T_e has a peak point. Its position $T_e = T_e^*$ depends on n. The height of the maximum $N = N^*$ increases with n. As it was shown in [11] by means of the linear stability analysis, only right branches of these curves correspond to the physically stable (realizable) states. A curve for $n = -1$ is valid for an uniform local heat flux over the whole heat transfer surface of the fin. For $-1 \leqslant n \leqslant 0$ all curves intersect the ordinate axis ($T_e = 0$) at definite value of $N = N^+$. For example, curves N vs T_e for $n = -0.75$ and $n = -0.5$ are seen to intersect the ordinate axis at $N^+ \simeq 2.1$ and 3.5, respectively. Forms of all curves for $n > 0$ are similar to a form of the well-known curve for $n = 0$. The parameter N goes to infinity at $T_e \to 0$ and all curves verge towards the ordinate axis at different N depending upon n. The more is n the greater value of N is reached by the curves near the ordinate axis. For $n = 0$

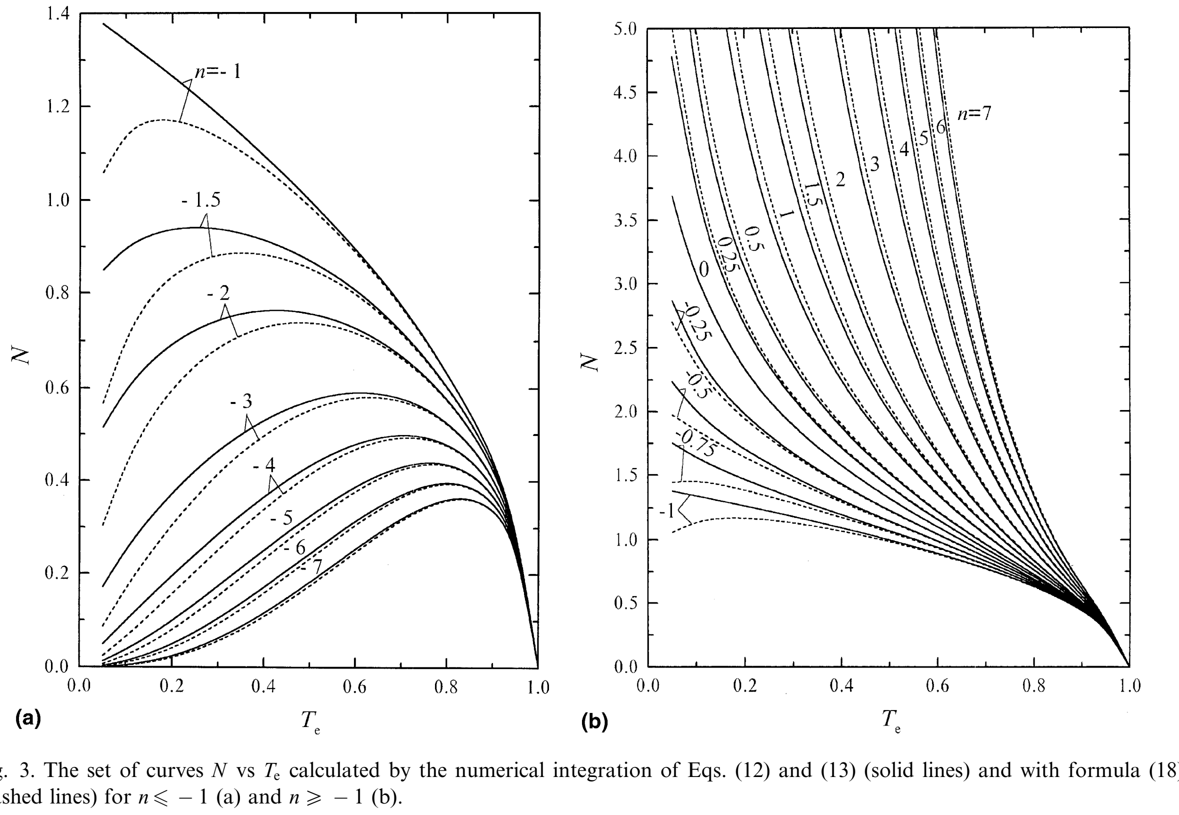

Fig. 3. The set of curves N vs T_e calculated by the numerical integration of Eqs. (12) and (13) (solid lines) and with formula (18) (dashed lines) for $n \leqslant -1$ (a) and $n \geqslant -1$ (b).

the corresponding value of N according to Eq. (17) (see below) for $T_e = 10^{-2}$ is equal to 5.3. At $n > 0$ the curves come towards the ordinate axis at still further increasing values of N.

3. Development of the simple closed-form formula

3.1. The basic idea of the development

Our first goal is to determine a closed-form expression for the dependence of the fin convective–conductive parameter N on the fin tip temperature difference T_e for a given exponent n in the range -7 to $+7$. How can this problem be solved for the whole range of the independent variables taking into account a complex non-linear character of the N vs T_e curves for given n shown in Fig. 1? The basic idea which enables one to achieve this task consists in the following. Let us consider how the quotient N/N_0 varies upon T_e at given n, where N_0 is an expression for N at $n = 0$, i.e., at the uniform heat transfer coefficient over the whole surface of the fin. This dependence for $T_e = 0.1 - 1$ is plotted by solid lines in Fig. 2 in log–log scale for negative and positive values of n using results of the numerical integration according to Eqs. (12) and (13). These curves prove to be much simple than curves in Fig. 1. All curves in Fig. 2 have relatively small curvature. Every curve contains a portion which is very close to a straight line tangent to it at $T_e = 1$. The latter are displayed in Fig. 2 by short dashed lines. The derivation of the exact general equation for these tangents will be presented in Appendix B. Its result has the following form:

$$N/N_0 = T_e^{-\mu n}, \tag{16}$$

where $\mu = 5/12$. According to Fig. 2, for both negative and positive n the straight short dashed lines are very close to the corresponding solid lines within the practically most important range $0.4 < T_e < 1$. The relative discrepancy between each curve and its tangent at $T_e = 1$ is sufficiently small at $0.4 \leqslant T_e \leqslant 1$ and increases when $T_e \to 0$.

It is known that the convective–conductive parameter N_0 for a fin with the uniform heat transfer coefficient ($n = 0$) can be expressed as

$$N_0 = \operatorname{arcosh}(1/T_e) = \ln\left[\left(1 + \sqrt{1 - T_e^2}\right)/T_e\right]. \tag{17}$$

Using Eqs. (16) and (17), we get the following closed-form relationship for the dependence of the fin parameter N on T_e and n:

$$N = T_e^{-\mu n}\operatorname{arcosh}(1/T_e) \tag{18}$$

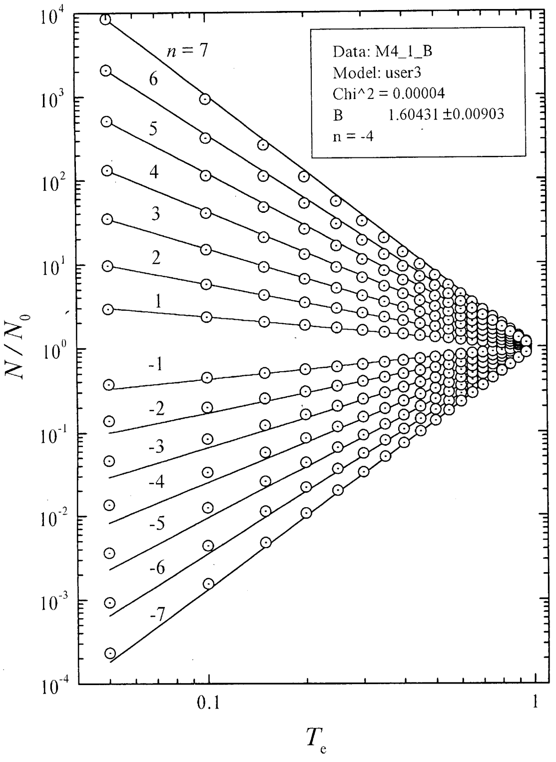

Fig. 4. Polynomial fit $\log(N/N_0)$ vs $\log(T_e)$ (solid lines) of data obtained by the numerical integration of Eqs. (12) and (13) (dot-centered open circles) for negative and positive values of n. The example of the fit coefficient for $n = -4$ is given in the insert to the figure.

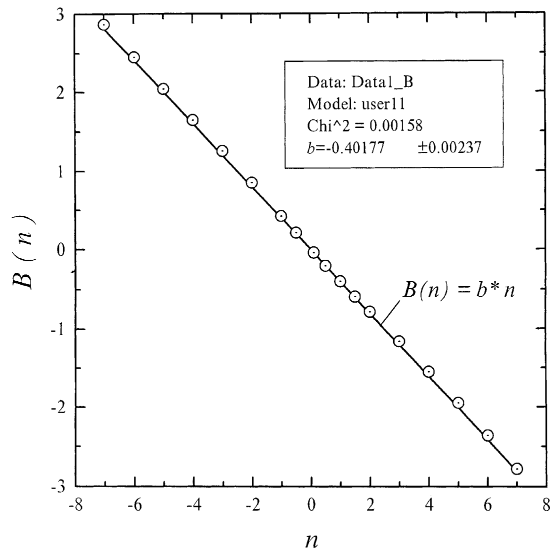

Fig. 5. Dependence of the polynomial coefficient B on the exponent n obtained using data of the numerical integration presented in Fig. 4 (dot-centered open circles) and their linear fit (solid straight line).

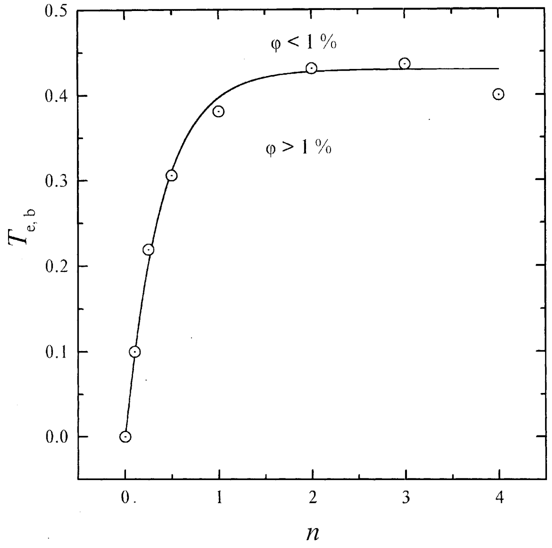

Fig. 6. Dependence of the threshold values of $T_{e,b}$ on the positive values of n (dot-centered open circles) and their exponential fit (solid curve).

or, equivalently,

$$N = T_e^{-\mu n} \ln\left[\left(1+\sqrt{1-T_e^2}\right)/T_e\right].$$

The comparison between the results calculated numerically and obtained by means of Eq. (18) is displayed in Fig. 3(a) for $-7 \leqslant n \leqslant -1$ and in Fig. 3(b) for $-1 \leqslant n \leqslant 7$ by solid and short dashed lines, respectively. It is clearly seen that all corresponding curves coincide not only qualitatively but also quantitatively in the wide and most important for practice range of independent variables. For $n < -1$ the best coincidence is observed on the right (physically realizable) branches of N vs T_e curves. For $n > -1$ results of evaluation using Eq. (18) are close to those obtained by numerical integration in a reasonably wide range of T_e. Thus, simple Eq. (18) expressed in terms of ordinary functions allows to describe a complex behavior of N vs T_e for given n. A noticeable discrepancy is observed for small T_e only.

To find a better correlation, we used also another approach for the analysis of the numerical data. We presented the data of the numerical evaluation of N/N_0 vs T_e in a form

$$\log(N/N_0) = B(n)\log T_e. \tag{19}$$

By means of the fit function procedure, which gives results shown in Fig. 4, we have defined coefficients B for different $-7 \leqslant n \leqslant 7$. An example of this type procedure for $n = -4$ is shown in the insert of Fig. 4. The dependence of B on n is displayed in Fig. 5. The curve $B(n)$ is seen to be a straight line and upon applying the analogous fitting procedure (results of the fitting are shown in the insert of Fig. 5) can be represented by the formula

$$B(n) = -0.4n. \tag{20}$$

The coefficient 0.4 in this equation is close to the coefficient $\mu = 5/12 = 0.4167$ obtained above. Below we shall use Eq. (18) with coefficient $\mu = 0.4$ in exponent instead of $\mu = 5/12$, taking into account that this value

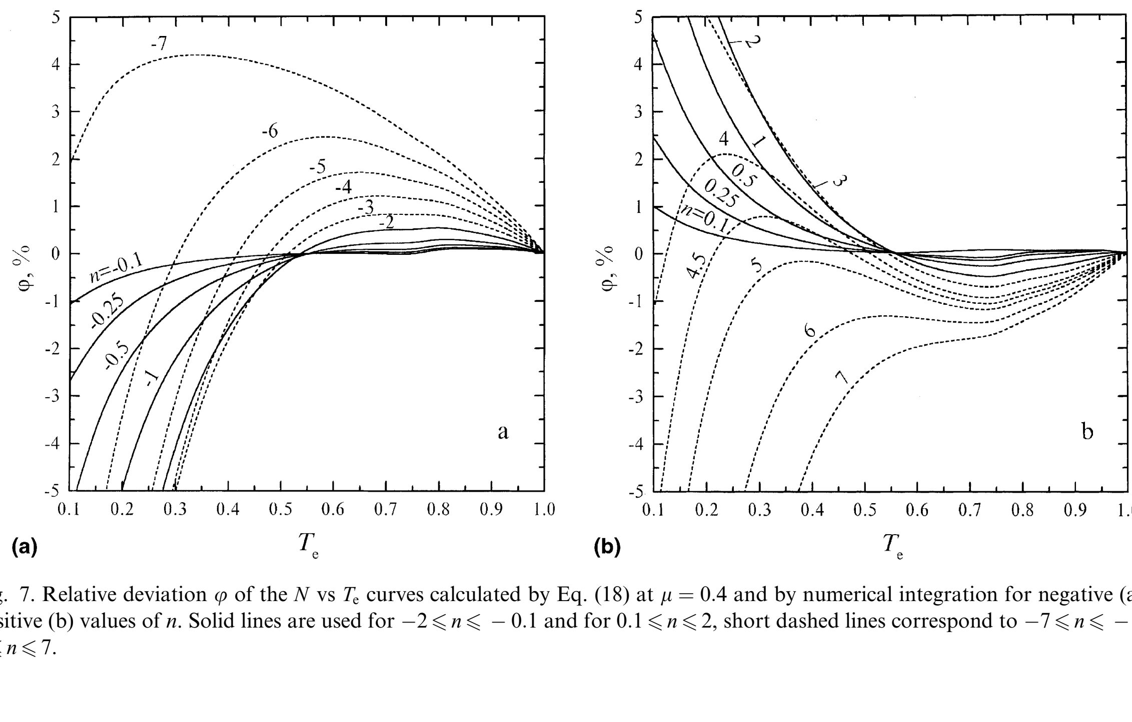

Fig. 7. Relative deviation φ of the N vs T_e curves calculated by Eq. (18) at $\mu = 0.4$ and by numerical integration for negative (a) and positive (b) values of n. Solid lines are used for $-2 \leqslant n \leqslant -0.1$ and for $0.1 \leqslant n \leqslant 2$, short dashed lines correspond to $-7 \leqslant n \leqslant -3$ and $3 \leqslant n \leqslant 7$.

gives more accurate results in a wider range of T_e. Using obtained results it is possible to find a threshold value $T_e = T_{e,b}$ whereby the discrepancy between approximate and "exact" (numerical) data does not exceed 1 % in the region $T_{e,b} \leqslant T_e \leqslant 1$. The results of this processing of the numerical data are plotted in Fig. 6 by the dot-centered open circles. Dependence of $T_{e,b}$ on n for $0 \leqslant n \leqslant 4$ has been correlated by the following expression:

$$T_{e,b} = 0.43[1 - \exp(-2.65n)]. \tag{21}$$

It is plotted in Fig. 6 by the solid line.

3.2. *The relative accuracy of the closed-form formula compared to the results of the numerical integration*

In this subsection we present the comparison between the results calculated with the help of the derived closed-form formula and using the numerical integration of Eq. (14). The relative deviation φ of the N vs T_e curves calculated with the use of formula (18) at $\mu = 0.4$ and by numerical integration of negative (a) and positive (b) values of n is plotted in Fig. 7. It is evident that dependence φ on T_e and n is rather complex. Hence solid lines are used for $-2 \leqslant n \leqslant -0.1$ and for $0.1 \leqslant n \leqslant 2$, short dash lines are used for $-7 \leqslant n \leqslant -3$ and for $3 \leqslant n \leqslant 7$, correspondingly. It can be seen that $\varphi < \pm 1\%$ holds in a wide range of T_e for both negative and positive n.

4. Conclusions

1. A generalized solution in terms of ordinary functions of the 1-D heat conduction problem for a single straight fin or spine of constant cross-section with heat transfer coefficient varying as a power-law function of excess temperature is obtained. The closed-form Eq. (18) is derived by two methods. The first method makes use of the derivative of the quotient N/N_0 with respect to T_e at $T_e \to 1$, where N and N_0 are the exact hypergeometric and well-known hyperbolic solutions, respectively. The second method uses fitting procedure for the function $\log(N/N_0)$ vs $\log T_e$ for given exponent n obtained as a result of the numerical integration of Eqs. (12) and (13).
2. The obtained formula at $n = 0$ transforms into the well-known expression for the fins with a uniform heat transfer coefficient.
3. The analytical evaluation of the fin parameter N calculated through the use of the derived closed-form formula Eq. (18) coincides well with numerical predictions in a wide range of n and T_e parameters.
4. The obtained closed-form formula in terms of ordinary functions allows to derive expression for direct evaluation of T_e in a wide range of given n and N. It can serve as a basis for the simple thermal design and optimization of the fins cooled by non-linear heat transfer mode (natural convection, boiling, radiation, etc.).
5. The fin base thermal conductance and fin augmentation factor (effectiveness) can be determined by the use of Eqs. (10) and (11) and the formula for T_e.

The last two items will be considered in the second part of the study [20].

Appendix A. A development of the analytical formula Eq. (15)

The integral in the LHS of Eq. (15) is taken by the following change of variable:

$$T = T_e + \xi. \tag{A.1}$$

Then

$$T^{n+2} - T_e^{n+2} \simeq [1 + (n+2)\xi/T_e]T_e^{n+2} - T_e^{n+2} = (n+2)T_e^{n+1}\xi. \tag{A.2}$$

Substituting Eq. (A.2) in the integrand of the integral equation (15) we get

$$\int_{T_e}^{T_e+\epsilon} \mathrm{d}T \Big/ \sqrt{[2/(n+2)](T^{n+2} - T_e^{n+2})} = \left(1/\sqrt{2T_e^{n+1}}\right) \int_0^{\epsilon} \mathrm{d}\xi/\sqrt{\xi} = \sqrt{2\epsilon/T_e^{n+1}}. \tag{A.3}$$

Appendix B. Analytical expression for derivative $\mathbf{d}(N/N_0)/\mathbf{d}T_e$ at $T_e \to 1$

For fixed constant n denote

$$f \equiv f(T_e) = N/N_0. \tag{B.1}$$

Accounting that N is determined by Eq. (12) for $n \neq -2$ and N_0 defined by Eq. (17) the derivative of the quotient N/N_0 with respect to T_e can be expressed as follows:

$$\frac{\mathrm{d}f}{\mathrm{d}T_e} = \frac{N}{N_0}\left[\frac{1}{N_0 T_e\sqrt{1 - T_e^2}} - \frac{n}{2T_e}\right] - \frac{\sqrt{(n+2)/2}}{N_0 T_e\sqrt{(1 - T_e^{n+2})}}. \tag{B.2}$$

Expand RHS of the above equation into the Taylor series at $T_e = 1$ with accuracy to the first-order terms, i.e.,

$$f \simeq 1 - (1 - T_e)\lim_{T_e \to 1}\frac{\mathrm{d}f}{\mathrm{d}T_e}. \tag{B.3}$$

Substituting this formula into Eq. (B.2) and solving the latter with respect to $\lim_{T_e \to 1}(\mathrm{d}f/\mathrm{d}T_e)$, we get

$$\lim_{T_e \to 1} \frac{\mathrm{d}f}{\mathrm{d}T_e} = \lim_{T_e \to 1} \Big\{ -n/(2T_e) + 1/\Big(T_e\sqrt{1-T_e^2}\Big) - 1/\Big(N_0 T_e \sqrt{1-T_e^{n+2}}\Big)\Big\} \Big/ \Big\{1 + (1-T_e) \times \Big[-n/(2T_e) + 1\Big/\Big(N_0 T_e \sqrt{1-T_e^2}\Big)\Big]\Big\}. \quad \text{(B.4)}$$

Expanding the numerator and the denominator of this equation in the series with respect to a small parameter $\epsilon = \sqrt{1-T_e^2}$, taking into account that according to Eq. (17)

$$N_0 \simeq \epsilon + \epsilon^3/3, \quad \text{(B.5)}$$

we get the following expression:

$$\lim_{T_e \to 1} \frac{\mathrm{d}f}{\mathrm{d}T_e} = \lim_{\epsilon \to 0} \frac{-n/2 + (\epsilon - n\epsilon^3/8 - \epsilon)/\epsilon^3}{1 + (\epsilon^2/2)(-n/2 + 1/\epsilon^2)} = \frac{-5n}{12}. \quad \text{(B.6)}$$

This limit is confirmed by the analytical mathematical program allowed to determine the derivative of the exact expression for N defined through the hypergeometric function (see introduction to this article). The limit according to Eq. (B.6) for $n = -2$ is equal 5/6. It is obtained by analogous procedure using Eqs. (13) and (17) for N and N_0.

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